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CONSTRUCTION OF MEASURES WITH EXPANSION

BY

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ABSTRACT

We give a construction of measures with partial sum of Lyapunov exponents bounded from below.

Introduction

Let M be a compact C^1-Riemannian manifold of dimension d and let $f\colon M \to M$ be a C^1-map. For $1 \leqslant k \leqslant d$, we denote by $\mathcal{S}_k$ the set of C^1-maps $\sigma\colon D^k = [0,1]^k \to M$. We define the k-volume of $\sigma \in \mathcal{S}_k$ by the formula:

$$V(\sigma) = \int_{D^k} |\Lambda^k T_x\sigma| d\lambda(x),$$

where $d\lambda$ is the Lebesgue measure on D^k and $|\Lambda^k T_x\sigma|$ is the norm of the linear map $\Lambda^k T_x\sigma\colon \Lambda^k T_x D^k \to \Lambda^k T_{\sigma(x)}M$ induced by the Riemannian metric on M. Some connections between the volume growth of iterates of submanifolds of M and the entropy of f have been studied by Y. Yomdin (see [8] and [4]), S. E. Newhouse (see [7]), O. S. Kozlovski (see [6]) and J. Buzzi (see [2]).

In this article, we prove that the volume growth of iterates of submanifolds of M permits to create invariant probability measures with partial sum of Lyapunov exponents bounded from below (see [5] for the definition of Lyapunov exponents). More precisely, for $1 \leqslant k \leqslant d$ we define the k-volume-expansion:

$$d_k := \limsup_{n\to\infty} \frac{1}{n} \log \sup_{\sigma\in\mathcal{S}_k} \frac{V(f^n\circ\sigma)}{V(\sigma)}.$$

We will prove the following theorem.

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THEOREM: *For all integer k between 1 and $d = \dim(M)$ there exists an ergodic invariant probability measure $\nu(k)$ for which*

$$\sum_{i=1}^{k} \chi_i \geqslant d_k.$$

Here $\chi_1 \geqslant \chi_2 \geqslant \cdots \geqslant \chi_d$ are the Lyapunov exponents of $\nu(k)$.

Notice that when $k = d$ and f is a ramified covering in some sense, the theorem can be deduced from a result due to T.-C. Dinh and N. Sibony (see [3] Paragraph 2.3).

Proof of the theorem

Let k be a positive integer between 1 and d. We have to prove that there exists an ergodic invariant probability measure $\nu(k)$ for which (see [1] Chapter 3)

$$\sum_{i=1}^{k} \chi_i = \lim_{m\to\infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(k)(y) \geqslant d_k. \tag{1}$$

There will be three steps in the proof of the theorem.

In the first one, we will change the volume-expansion d_k into an expansion of $|\Lambda^k T_x f^n|$. More precisely, we will find points x_{n_l} with $\frac{1}{n_l} \log |\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant d_k - \varepsilon(l)$ (with $\varepsilon(l) \to 0$ when $l \to +\infty$).

In the second part, we will see that the expansion of $|\Lambda^k T_{x(n_l)} f^{n_l}|$ can be spread out in time. We will give the construction of a measure ν_l such that $d_k - \varepsilon'(l) \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y)$ (with $\varepsilon'(l) \to 0$ when $l \to +\infty$).

The third step of the proof will be to take the limit when $l \to \infty$ in the previous inequality.

1) FIRST STEP. Let n_l be a subsequence such that

$$\frac{1}{n_l} \log \sup_{\sigma \in \mathcal{S}_k} \frac{V(f^{n_l} \circ \sigma)}{V(\sigma)} \to d_k.$$

We can find now a sequence $\sigma_{n_l} \in \mathcal{S}_k$ which verifies

$$\frac{1}{n_l} \log \frac{V(f^{n_l} \circ \sigma_{n_l})}{V(\sigma_{n_l})} \to d_k.$$

In the next lemma, we prove that we have expansion for $|\Lambda^k T_x f^n|$ for some x.

LEMMA 1: *For all $l \geqslant 0$ there exists $x(n_l) \in M$ with:*

$$\log|\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant \log\Big(\frac{V(f^{n_l}\circ\sigma_{n_l})}{2V(\sigma_{n_l})}\Big).$$

Proof: If this was not the case, then we would have an integer l such that for all $x \in M$,

$$|\Lambda^k T_x f^{n_l}| \leqslant \frac{V(f^{n_l}\circ\sigma_{n_l})}{2V(\sigma_{n_l})}.$$

So (see [1] chapter 3.2.3 for properties on exterior powers),

$$V(f^{n_l}\circ\sigma_{n_l}) = \int_{D^k} |\Lambda^k T_x(f^{n_l}\circ\sigma_{n_l})| d\lambda(x) = \int_{D^k} |\Lambda^k T_{\sigma_{n_l}(x)} f^{n_l} \circ \Lambda^k T_x \sigma_{n_l}| d\lambda(x)$$

is bounded from above by

$$\int_{D^k} |\Lambda^k T_{\sigma_{n_l}(x)} f^{n_l}||\Lambda^k T_x\sigma_{n_l})| d\lambda(x) \leqslant \frac{V(f^{n_l}\circ\sigma_{n_l})}{2}$$

and we obtain a contradiction. ∎

COROLLARY 1: *There exists a sequence $\varepsilon(l)$ which converges to 0 such that*

$$\frac{1}{n_l}\log|\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant d_k - \varepsilon(l),$$

for some points $x(n_l)$ in M.

2) SECOND STEP. In this section, we will spread out in time the previous expansion. Let m be a positive integer. We will cut now n_l in m different ways.

By using the Euclidian division, we can find q_l^i and r_l^i (for $i = 0, \ldots, m-1$) such that

$$n_l = i + m \times q_l^i + r_l^i,$$

with $0 \leqslant r_l^i < m$.

If $i \in \{0, \ldots, m-1\}$, we have

$$|\Lambda^k T_{x(n_l)} f^{n_l}| \leqslant |\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}| \times \prod_{j=0}^{q_l^i - 1} |\Lambda^k T_{f^{i+jm}(x(n_l))} f^m| \times |\Lambda^k T_{x(n_l)} f^i|,$$

so, using the previous corollary,

$$n_l(d_k - \varepsilon(l)) \leqslant \log|\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}| + \sum_{j=0}^{q_l^i-1} \log|\Lambda^k T_{f^{i+jm}(x(n_l))} f^m| + \log|\Lambda^k T_{x(n_l)} f^i|.$$

If we sum over the m different ways to write n_l, we obtain:

$$mn_l(d_k-\varepsilon(l))\leqslant\sum_{i=0}^{m-1}\log|\Lambda^k T_{f^{i+mq_l^i}(x(n_l))}f^{r_l^i}|+\sum_{i=0}^{m-1}\sum_{j=0}^{q_l^i-1}\log|\Lambda^k T_{f^{i+jm}(x(n_l))}f^m| + \sum_{i=0}^{m-1}\log|\Lambda^k T_{x(n_l)}f^i|.$$

We have to transform this estimate into a property of a measure. To achieve this we remark that

$$\log|\Lambda^k T_{f^p(x(n_l))}f^m| = \int \log|\Lambda^k T_y f^m| d\delta_{f^p(x(n_l))}(y),$$

where $\delta_{f^p(x(n_l))}$ is the dirac measure at the point $f^p(x(n_l))$.

So the previous inequality becomes

$$d_k-\varepsilon(l)\leqslant a_l+\frac{1}{m}\int\log|\Lambda^k T_y f^m| d\Big(\frac{1}{n_l}\sum_{i=0}^{m-1}\sum_{j=0}^{q_l^i-1}\delta_{f^{i+mj}(x(n_l))}\Big)(y)+b_l,$$

with

$$a_l=\frac{1}{mn_l}\sum_{i=0}^{m-1}\log|\Lambda^k T_{f^{i+mq_l^i}(x(n_l))}f^{r_l^i}|$$

and

$$b_l=\frac{1}{mn_l}\sum_{i=0}^{m-1}\log|\Lambda^k T_{x(n_l)}f^i|.$$

Now, since f is a C^1-map we have:

$$a_l\leqslant\frac{1}{mn_l}\sum_{i=0}^{m-1}\log L^{mk}\leqslant\frac{km^2}{mn_l}\log L$$

where $L=\max(\max_x|T_x f|,1)$ and

$$b_l\leqslant\frac{1}{mn_l}\sum_{i=0}^{m-1}\log L^{mk}\leqslant\frac{km^2}{mn_l}\log L.$$

So the sequences a_l and b_l are bounded from above by a sequence which converges to 0 when l tends to infinity.

In conclusion, we have

$$d_k-\varepsilon'(l)\leqslant\frac{1}{m}\int\log|\Lambda^k T_y f^m| d\nu_l(y), \tag{2}$$

with

$$\nu_l = \frac{1}{n_l} \sum_{i=0}^{m-1} \sum_{j=0}^{q_l^i - 1} \delta_{f^{i+mj}(x(n_l))},$$

and $\varepsilon'(l)$ a sequence which converges to 0.

3) Third step. The aim of this section is to take a limit in (2).

First, observe that

$$\nu_l = \frac{1}{n_l} \sum_{p=0}^{n_l - m} \delta_{f^p(x(n_l))}$$

and that the sequence $\frac{1}{n_l} \sum_{p=0}^{n_l-1} \delta_{f^p(x(n_l))} - \nu_l$ converges to 0. In particular, there exists a subsequence of ν_l which converges to a measure ν which is a probability measure invariant under f and independent of m. We continue to call ν_l the subsequence which converges to ν. To complete the proof of the theorem, we have to take the limit in (2). However, we have to be careful because the function $y \mapsto \log |\Lambda^k T_y f^m|$ is not continuous. But, we have the following lemma.

Lemma 2:

$$\limsup_{l \to \infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(y).$$

Proof: For $r \in \mathbb{N}$, let $\Phi_r(y) = \max(\log |\Lambda^k T_y f^m|, -r)$. The functions Φ_r are continuous and the sequence Φ_r decreases to the map $y \mapsto \log |\Lambda^k T_y f^m|$ when r goes to infinity. Then

$$\frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \Phi_r(y) d\nu_l(y)$$

and

$$\limsup_{l \to \infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \Phi_r(y) d\nu(y),$$

because Φ_r is continuous. Now, we obtain the lemma by using the monotone convergence theorem. ∎

Remark: This lemma is valid for any upper semicontinuous function instead of $y \mapsto \frac{1}{m} \log |\Lambda^k T_y f^m|$.

It remains to take the limit when $l \to +\infty$ in (2). Then we obtain

Corollary 2: *For all m, we have*

$$d_k \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(y).$$

In particular (see (1)),

$$d_k \leqslant \int \sum_{i=1}^{k} \chi_i(y) d\nu(y)$$

where the $\chi_1 \geqslant \chi_2 \geqslant \cdots \geqslant \chi_d$ are the Lyapunov exponents of ν. Finally, by using the ergodic decomposition of ν, we obtain the existence of an ergodic invariant probability measure $\nu(k)$ with

$$d_k \leqslant \sum_{i=1}^{k} \chi_i.$$

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